PHYSICAL REVIEW E **94**, 052310 (2016)

Robustness of controlling edge dynamics in complex networks against node failure

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(Received 14 July 2016; revised manuscript received 26 August 2016; published 14 November 2016)

The robustness of controlling complex networks is significant in network science. In this paper, we focus on evaluating and analyzing the robustness of controlling edge dynamics in complex networks against node failure. Using three categories of all nodes to quantify the robustness, we find that the percentages of the three types of nodes are mainly related to the degree distribution of networks. The simulation results of model networks and analytic calculations show that the sparse inhomogeneous networks, which emerge in many real complex networks, have strong control robustness from the point of the number of ordinary nodes, but the strong positive correlation between in and out degrees reduces the control robustness. Evaluation of real-world networks indicates that most of them have few or no critical nodes, that is, they do not need to increase driver nodes to maintain control for most of node failures. Then an adding circuit-link strategy is proposed to optimize the robustness of edge controllability.

DOI: 10.1103/PhysRevE.94.052310

I. INTRODUCTION

Complex networks, composed of interacting individual units, universally exist in most technological, physical, social, and biological systems [1–4]. How to control complex networks is a fundamental issue in contemporary network science [5–7]. The key issue addressed by the control theories is, with a suitable choice of inputs, to steer the network system to any desired final state in finite time [8,9]. Liu *et al.* [10] made a breakthrough by developing a structural control theory for nodal dynamics of complex networks and offered efficient tools based on the maximum matching to determine the minimum number of driver nodes. Then Nepusz and Vicsek [11] analyzed edge dynamics in complex networks and found that the controllability property of edge dynamics significantly differs from that of nodal dynamics. Much interest has been stimulated toward exploring the controllability properties of complex networks [12–17].

The robustness of controlling complex networks, which represents the property of being strong and healthy in control, is significant in network science. Generally, uncertain failures and perturbations are inevitable in real-world complex networks. The network structure always confronts random failures and intentional attacks. So the robustness [18], the ability to withstand failures and perturbations, has been one of the most active topics in network science. In particular, robustness is crucial for the infrastructure networks such as power grids [19], the Internet [20], transportation networks [21], etc. Evidence has demonstrated that such networks can be affected by failures and attacks that emerge locally. Liu *et al.* [10] analyzed the robustness of the nodal controllability in networks under unavoidable node (edge) failure. They classified nodes (edges) into three categories: critical, redundant, and ordinary. They found that most nodes (edges) are ordinary in real-world networks, and proved that the content of cores and leaves in a network is the key factor determining the proportion of three node (edge) categories. The robustness of controlling edge dynamics of networks is another crucial issue. Nepusz and Vicsek [11] similarly classified edges into three categories. They found that most real-world networks have strong robustness of control against edge failure, and showed that the category of an edge can be determined by its local information, i.e., the in and out degrees of its two endpoints.

Despite the significant findings, evaluating and analyzing the robustness of edge controllability of complex networks against node failure are still lacking. In this paper, we focus on this issue, which could help deepen our understanding of the robustness of controlling complex networks against uncertain failures and perturbations. In a way similar to that in Refs. [10,11], each node is classified by the change in the number of driver nodes when the node and its links are removed from networks. To be specific, for a given network, the number of driver nodes is denoted by N_{D}. After a node and its links are removed, the number of driver nodes in the remainder network is denoted by N'_{D}. Nodes can be classified into three categories: critical, redundant, and ordinary. The removal of a critical node increases the number of driver nodes required to maintain full controllability, i.e., $N'_{\mathrm{D}} > N_{\mathrm{D}}$. Conversely, the removal of a redundant node decreases the number of driver nodes, i.e., $N'_{\mathrm{D}} < N_{\mathrm{D}}$. The rest nodes are ordinary since removing them does not affect the number of driver nodes, i.e., $N'_{\mathrm{D}} = N_{\mathrm{D}}$. For instance, in Fig. 1, node a is redundant, node b is critical, and the rest of the nodes are ordinary.

This classification leads to the quantitative analysis of the robustness of edge controllability in complex networks against node failure. We find that, unlike that of the robustness of controlling node dynamics, the percentages of the three node categories are mainly related to the degree distribution of the network. The simulation and theoretical analysis for model networks show that sparse inhomogeneous networks have

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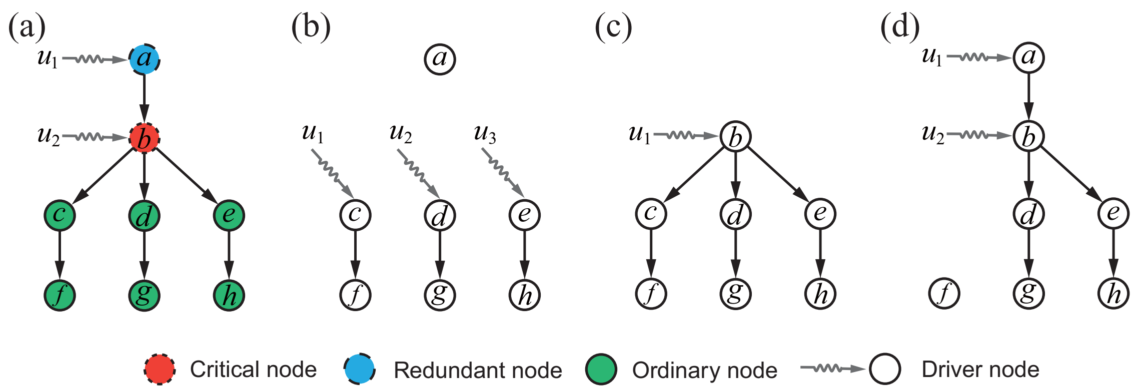

FIG. 1. Categories of nodes in a network. (a) A network with eight nodes that can be control via two driver nodes ($N_{\mathrm{D}} = 2$). Node a is redundant, node b is critical, and the rest of the nodes are ordinary. (b) The removal of critical node b increases the number of driver nodes ($N'_{\mathrm{D}} = 3$). (c) The removal of redundant node a decreases the number of driver nodes ($N'_{\mathrm{D}} = 1$). (d) The removal of ordinary node c does not affect the number of driver nodes ($N'_{\mathrm{D}} = 2$).

strong control robustness, but the strong positive correlation between in and out degrees reduces the control robustness. Evaluation of real-world networks indicates that most of them, particularly the transcriptional regulatory networks, have few or no critical nodes, i.e., they do not need to increase driver nodes to maintain control for most of node failures. Finally, an adding circuit-link strategy is proposed to optimize the robustness of controlling edge dynamics in any network.

II. NODE CATEGORY

A. Switchboard dynamics

The switchboard dynamics was proposed in Ref. [11] to characterize a dynamical process on the edges of a network $G(V,E)$. Specifically, each state variable corresponds to an edge of the network. The state variables of the outgoing edges of a node can be influenced by the state variables of the incoming edges of the node, the damping terms of the outgoing edges themselves, and the external inputs. The switchboard dynamics provides a simplified representation of the underlying dynamic processes in many real networks and gives rise to several conclusions of the structural controllability of edge dynamics that remarkably differ from nodal dynamics. The key result is that the minimum set of driver nodes, which is required to maintain structural controllability of the switchboard dynamics on a network, can be determined by selecting the divergent nodes ($k_v^+ > k_v^-$) and one arbitrary node from each balanced component ($k_v^+ = k_v^- > 0$ for all nodes in a connected component).

The switchboard dynamics is suitable for modeling networks where nodes are active components with information processing capabilities. We focus on the robustness of edge controllability under unavoidable node failure. For instance, consider the transport network where a node (i.e., a transfer station) constantly processes the passengers and goods received from its upstream neighbors and transfers them to the downstream neighbors. The passengers and goods received and passed by a node can then be represented by the state variables on its incoming and outgoing edges. A paralyzed transfer station, which corresponds to a removed node, will influence the transport of the passengers and goods in its upstream and downstream stations. Technological networks also be suitable candidates. Consider an internet network with computers and routers, where the edges represent physical connections. The amount of packet flow along a connection in a given direction can then be represented by the state variables. Each node is a router and the switching matrix may then represent a routing mechanism that allows packets to reach their destination while avoiding congestion. The failure of a router will influence the transport of packets in the network.

B. Identifying node category

In general, the balanced component, which is infrequent in directed networks, has little influence to the number N_{D} of driver node [11]. We thus neglect the possible presence of balanced components. Then the classification (driver or nondriver) of a node depends solely on its local information, i.e., the in and out degrees of the node. A node v is called weakly divergent if its out degree is one larger than its in degree, i.e., $k_v^+ = k_v^- + 1$, and is balanced if it has the same in and out degrees, i.e., $k_v^+ = k_v^-$. If a weakly divergent node appears in the upstream neighbors of the removed node, the weakly divergent node will turn into a nondriver one. Conversely, if a balanced node appears in the downstream neighbors of the removed node, the balanced node will turn into a driver one. Note that N'_{D} will drop an additional one if the removed node is divergent itself. We can therefore distinguish three cases:

(1) A divergent node v is critical if the number of balanced nodes N_v^{B} in its downstream neighbors and the number of weakly divergent nodes N_v^{W} in its upstream neighbors satisfy $N_v^{\mathrm{B}} > N_v^{\mathrm{W}} + 1$. A nondivergent node v is critical if $N_v^{\mathrm{B}} > N_v^{\mathrm{W}}$.

(2) A divergent node v is redundant if the number of balanced nodes N_v^{B} in its downstream neighbors and the number of weakly divergent nodes N_v^{W} in its upstream neighbors satisfy $N_v^{\mathrm{B}} \leqslant N_v^{\mathrm{W}}$. A nondivergent node v is redundant if $N_v^{\mathrm{B}} < N_v^{\mathrm{W}}$.

(3) In all other cases, the node is ordinary.

Note that the upstream and downstream neighbors of a node do not contain the node itself when a loop exists in this node.

III. ROBUSTNESS OF CONTROL

Three categories of all nodes could quantify the robustness of edge controllability against node failure. A network with more ordinary nodes has stronger control robustness. In this section, we will analyze the factors that mainly determine the proportion of three node categories in a network. The numerical simulations and theoretical analysis will be performed on the model networks including Erdős-Rényi (ER) random network [22], exponential (EX) network and scale-free (SF) network [23], and real-world networks (the detailed information of real-world networks is shown in Table I).

A. Model networks

A model network with N nodes is structured by the same given in- and out-degree distributions (Poisson distribution, exponent distribution, or power-law distribution), i.e., $P_k^{\rm in} = P_k^{\rm out} = P_k$. One can obtain the degree sequence by P_k, where the in- and out-degree sequences are denoted by $K_{\rm in} = \{k_1^-, k_2^-, \ldots, k_N^-\}$ and $K_{\rm out} = \{k_1^+, k_2^+, \ldots, k_N^+\}$, respectively, for convenience of discussions. Note that the in-degree sequence is the same as the out-degree sequence, i.e., $k_i^- = k_i^+$ $(i = 1,2,\ldots,N)$. The directed network starts from N isolated nodes. Each node is randomly assigned in degree k_i^- and out degree k_j^+ from in- and out-degree sequences, respectively. Then two nodes u with $k_u^- > 0$ and node v with $k_v^+ > 0$ are randomly selected and connected with direction from node v to u. Then the in degree of node u turns into $k_u^- - 1$ and the out degree of node v turns into $k_v^+ - 1$. This process is repeated until all nodes satisfy the given in and out degrees. Note that the multiple edges in the generated network will be disposed by edges exchanging, i.e., turning edges e_{uv} and e_{kl} to edges e_{ul} and e_{kv} if there exist multiple edges e_{uv}, where $k \neq u$ and $l \neq v$.

B. Robustness of control in model networks

We will analyze which factor will mainly determine the proportion of three node categories in a network. For this, we recall the density of critical, redundant, and ordinary nodes denoted by $n_{\rm crit} = N_{\rm crit}/N$, $n_{\rm red} = N_{\rm red}/N$, and $n_{\rm ord} = N_{\rm ord}/N$, respectively. In Fig. 2, we show their average degree $\langle k \rangle$ dependence for ER network and EX network and scale-free exponent γ dependence for SF network.

The simulation and analysis results for ER network with different average degrees $\langle k \rangle$ are shown in Fig. 2(a). One can see that at low average degrees, the network is dominated by ordinary nodes. The reason is that abundant isolated nodes are existing in the network. Also, the removal of isolated nodes

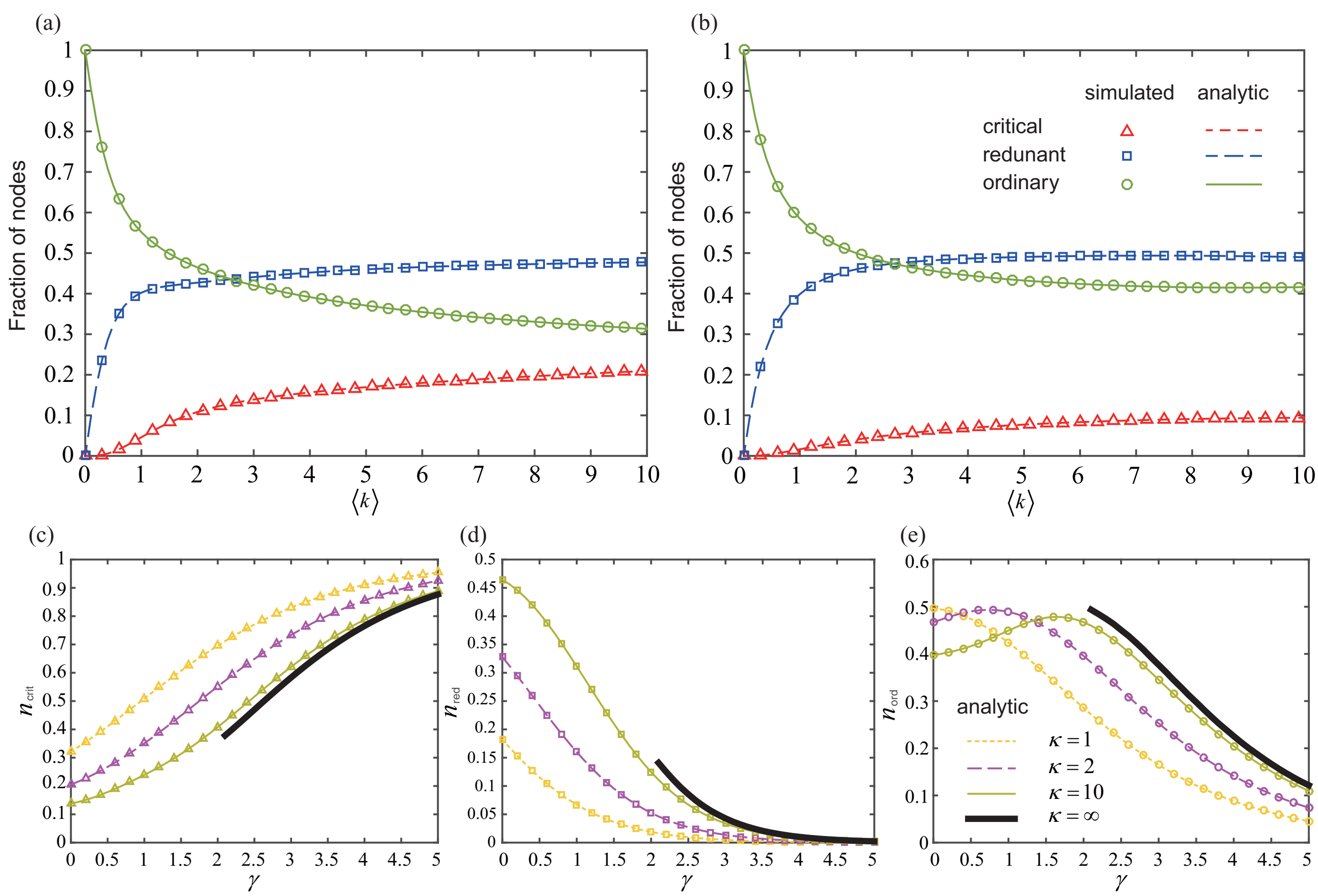

FIG. 2. Fraction of nodes. Fraction of critical (triangle), redundant (square), and ordinary (circle) nodes in ER network (a) and EX network (b) as the function of the average degree $\langle k \rangle$, and in SF network [(c), (d), and (e)] as the function of the scale-free exponent γ and the exponential cutoff parameter κ. All the simulation results are obtained by averaging over 20 independent network realizations.

has no effect on N_D. As the network grows, the fraction of redundant nodes increases rapidly. In this range, there are many small disconnected components containing two nodes and one edge only in the network. For these components, both nodes are redundant since the removal of one of them can reduce N_D. For EX network in Fig. 2(b), the trends of the curves are very similar to those provided from ER network. But the fraction of critical nodes is smaller, counterbalanced by a greater proportion of ordinary nodes.

Figures 2(c), 2(d), and 2(e) show simulation and analysis results for SF network with different scale-free exponent γ and exponential cutoff parameter κ. One can see that the SF network is dominated by critical nodes for large γ. Note that the average degree depends on both γ and κ as $\langle k\rangle = \mathrm{Li}_{\gamma-1}(e^{-1/\kappa})/\mathrm{Li}_{\gamma}(e^{-1/\kappa})$, where $\langle k\rangle$ increases with the increase of κ, while it decreases with the increase of γ. When $\gamma > 3$, nodes with only one incoming edge and only one outgoing edge appear with very high probability. So the removal of a node may converts its downstream node to a driver node with one outgoing edge and no incoming edges. Another interesting phenomenon is that the fraction of ordinary nodes shows a distinct peak for a specific value of γ. Both the location and the height of the peak depend on κ, and larger value of κ move the peak to the direction where γ increases.

The simulation and analysis results show that the sparse and inhomogeneous ($\gamma < 2$) networks have strong control robustness from the point of the number of ordinary nodes. However, the dense inhomogeneous networks have strong control robustness against edge failure [11]. The main reason for different results is that the classification of node and edge is different. Specifically, the classification of a node depends on the changes of its upstream and downstream neighbors when the node and its links are removed, while the classification of an edge depends solely on the in and out degrees of its endpoints. The sparse network contains abundant isolated nodes and disconnected components containing two nodes and one edge. The removal of isolated nodes has no effect on N_D, while the only edge in each component is redundant by the definition since its removal reduces N_D. So the sparse network is dominated by ordinary nodes, but has a relatively small percentage of ordinary edges. Conversely, the dense network is dominated by ordinary edges since it contains few balanced and weakly divergent nodes [11]. Meanwhile, the dense network has more redundant nodes than ordinary nodes since the removal of a divergent node itself can reduce N_D. Furthermore, the homogeneous network contains many balanced nodes, and the remove of any one incoming edge of the balanced node can increase N_D. Thus, compared with the homogeneous network, the inhomogeneous network has stronger control robustness from the point of the number of ordinary nodes and ordinary edges.

C. Robustness of control in real-world networks

To analyze the robustness of edge controllability under unavoidable node failure in real-world networks, we give the fraction of critical, redundant, and ordinary nodes in each real-world network. As shown in Fig. 3(a), most networks have few or no critical nodes, meaning that they do not need to increase driver nodes to maintain control for most of node failures. A notable finding is that the fraction of redundant nodes is significantly higher than that of critical nodes. The reason is that a removed node may be a driver node itself, which increases the probability of the redundant nodes. This phenomenon demonstrates that most real-world networks are dominated by ordinary and redundant nodes.

Next we focus on the dependence of the fraction of critical and ordinary nodes in real-world networks. As shown in Fig. 3(b), n_{crit} and n_{ord} of real-world networks with different $\langle k\rangle$ indeed disperse in two relatively small regions, where the curves in two regions are analytical results of the fraction of critical and ordinary nodes in EX network, respectively. The dispersion of n_{crit} and n_{ord} in two regions is due to the influence from the network topology, or from the degree distribution. So we apply a degree-preserving randomization [24], which keeps the in and out degrees of each node unchanged but reconnects the nodes randomly. As shown in Fig. 3(c), this procedure does not alter the number of critical and ordinary nodes significantly. In conclusion, we find that the control robustness is, to a great extent, encoded by the degree distribution of network. Note that we just show the results of critical and ordinary nodes since $n_{\mathrm{crit}} + n_{\mathrm{ord}} + n_{\mathrm{red}} = 1$.

D. Disturbance strength

The removal of a node may disturb the number N_D of driver nodes required to maintain full controllability of the edge dynamics in networks. We use the difference $\delta = |N_D - N_D'|$ to quantify the disturbance strength that a removed node gives rise to. The densities of critical and redundant nodes with disturbance strength δ are denoted by $n_{\mathrm{crit}}^{\delta} = N_{\mathrm{crit}}^{\delta}/N$ and $n_{\mathrm{red}}^{\delta} = N_{\mathrm{red}}^{\delta}/N$, respectively. Then the average disturbance strength $\bar{\delta} = \sum_{i=1}^{N}(\delta n_{\mathrm{crit}}^{\delta} + \delta n_{\mathrm{red}}^{\delta})$ is used to quantify the fragility of a network; i.e., the controllability of a network with larger $\bar{\delta}$ is easier to be disturbed by node failure. In Figs. 4(a)–4(i), we show their average degree $\langle k\rangle$ dependence for ER and EX networks and scale-free exponent γ dependence for the SF network.

As shown in Figs. 4(a)–4(f), a significant result is that the critical and redundant nodes with $\delta = 1$ play a primary role, and the nodes with stronger disturbance strength have less probability of existence. This indicates that the disturbance strength of most nodes is not huge, and the number of nodes with stronger disturbance strength is small in networks. Another interesting phenomenon is that, in Fig. 4(c), a polarization appears between the critical nodes with $\delta = 1$ and other critical nodes in the SF network. The reason is that, in the SF network, the nodes with only one incoming edge and only one outgoing edge appear with very high probability when $\gamma > 3$. So the disturbance strength of most nodes is $\delta = 1$. As shown in Figs. 4(g)–4(i), the sparse inhomogeneous networks have lower $\bar{\delta}$. The simulation results reconfirm that the sparse inhomogeneous networks have strong control robustness against node failure.

Furthermore, to analyze the correlation between the disturbance strength of nodes and their degrees, the average degree of critical nodes with the same disturbance strengths δ is defined as $\langle k\rangle_{\mathrm{crit}}^{\delta}$, and $\langle k\rangle_{\mathrm{red}}^{\delta}$ for redundant nodes is defined similarly. In Figs. 4(j)–4(l), we show their δ dependence for model networks. One can see that, for all the model

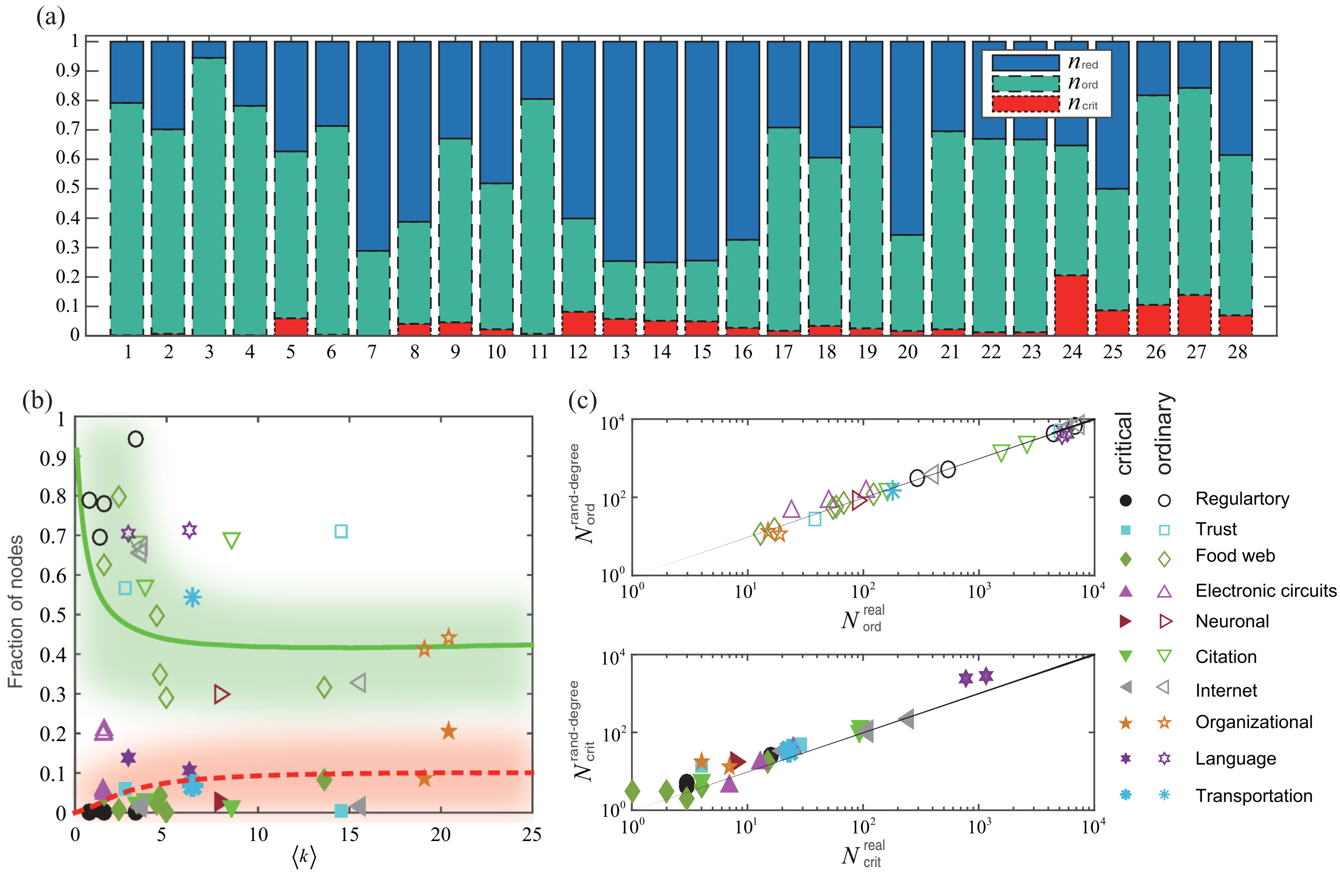

FIG. 3. Real-world networks. (a) Fraction of critical, ordinary, and redundant nodes in real-world networks. Numbers refer to the network indices in Table I. (b) Fraction of critical and ordinary nodes of real-world networks as the function of the average degree $\langle k \rangle$. The curves are analytical results of the fraction of critical (dashed red line) and ordinary (solid green line) nodes in EX network. (c) Number of critical and ordinary nodes, $N_{crit}^{rand-degree}$ and $N_{ord}^{rand-degree}$, obtained for the degree-preserving randomized version of the real-world networks, compared with N_{crit}^{real} and N_{ord}^{tral}, respectively.

networks, the obviously positive correlations exist between $\langle k\rangle_{crit}^{\delta}$ and δ and between $\langle k\rangle_{red}^{\delta}$ and δ. The simulation results indicate that the hubs tend to have greater disturbance strength.

E. Analytical results

For model networks, the way of identifying node category can be used to derive analytical formulas for the expected fraction of three node categories. To be specific, the category of a node is determined by the in and out degrees of itself, the number of weakly divergent nodes in its upstream neighbors, and the number of balanced nodes in its downstream neighbors. By enumerating all possible combinations that satisfied the conditions of critical, redundant, or ordinary nodes, we can offer the analytical formulas of their expected fractions. Note that the analytical results are offered by assuming that the in and out degrees of each node are uncorrelated in the model networks. The probability of finding a weakly divergent node is given by $P_d = \sum_{k=0}^{\infty} P_k P_{(k+1)}$, and the probability of finding a balanced node is given by $P_b = \sum_{k=1}^{\infty} P_k P_k$. By neglecting the possible existence of balanced components, the expected fraction of critical nodes can be described by

$$\begin{aligned} n_{crit} = & \sum_{k^-=0}^{\infty} \sum_{k^+=k^-+1}^{\infty} P_{k^-} P_{k^+} \left[\sum_{i=0}^{k^-} C_{k^-}^i P_d^i (1-P_d)^{k^- - i} \left(\sum_{j=i+2}^{k^+} C_{k^+}^j P_b^j (1-P_b)^{k^+ - j} \right) \right] \\ & + \sum_{k^-=k^+=1}^{\infty} P_{k^-} P_{k^+} \left[\sum_{i=0}^{k^- -1} C_{k^-}^i P_d^i (1-P_d)^{k^- - i} \left(\sum_{j=i+1}^{k^+} C_{k^+}^j P_b^j (1-P_b)^{k^+ - j} \right) \right] \\ & + \sum_{k^+=1}^{\infty} \sum_{k^-=k^++1}^{\infty} P_{k^-} P_{k^+} \left[\sum_{i=1}^{k^+} C_{k^+}^i P_b^i (1-P_b)^{k^+ - i} \left(\sum_{j=0}^{i-1} C_{k^-}^j P_d^j (1-P_d)^{k^- - j} \right) \right], \end{aligned} \quad (1)$$

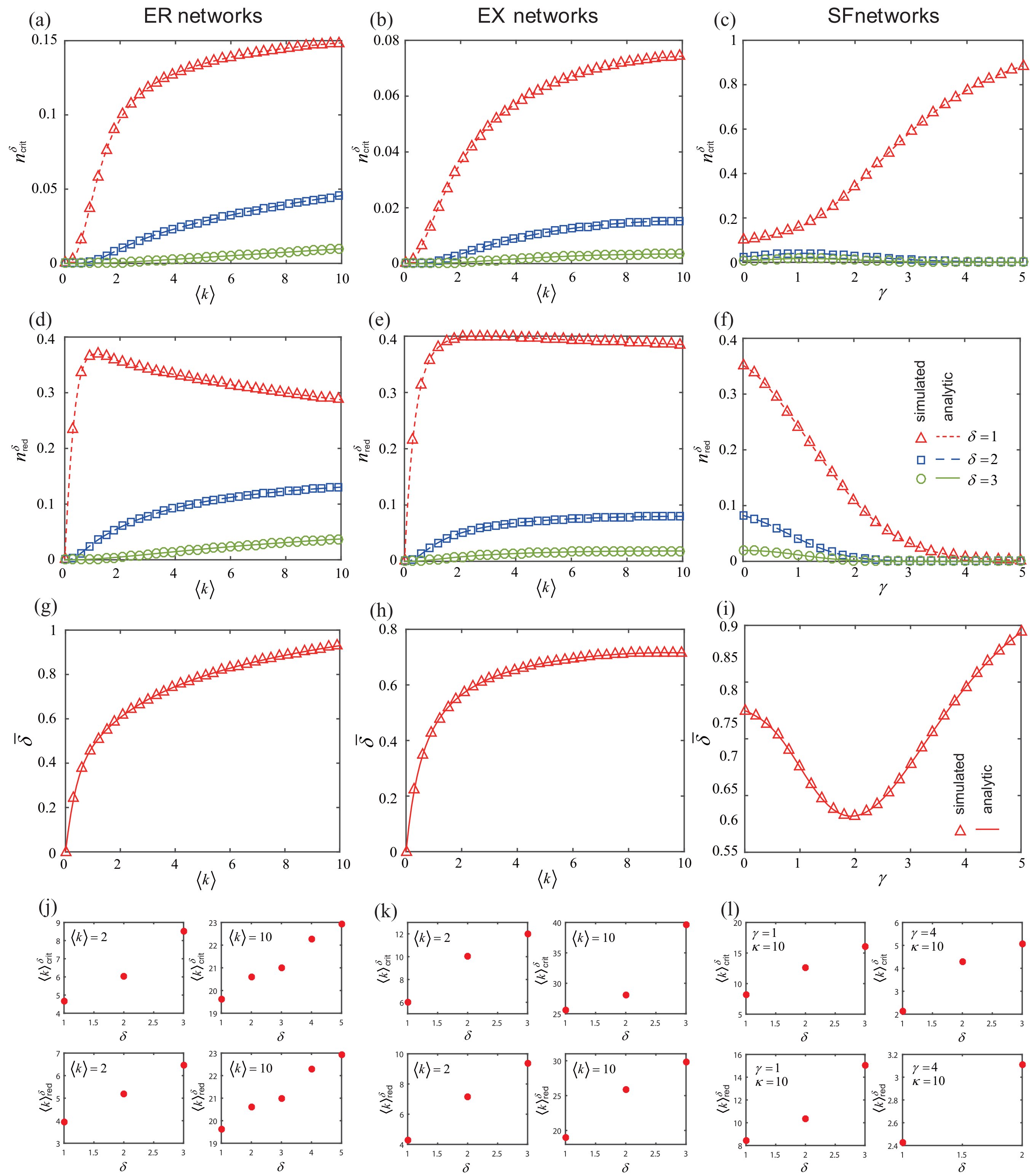

FIG. 4. Disturbance strength. Fraction of critical and redundant nodes with different disturbance strengths in ER network [(a), (d)] and EX network [(b), (e)] as the function of the average degree $\langle k \rangle$, and in SF network [(c), (f)] with the exponential cutoff parameter $\kappa = 10$ as the function of the scale-free exponent γ. The average disturbance strength $\bar{\delta}$ in ER network (g) and EX network (h) as the function of $\langle k \rangle$, and in SF network (i) with $\kappa = 10$ as the function of γ. The average degree of critical and redundant nodes with the same disturbance strengths in ER network (j), EX network (k), and SF network (l) as the function of δ. All the simulation results are obtained by averaging over 20 independent networks realizations.

where the first line is the case of divergent nodes and the rest of the lines are the cases of nondivergent nodes. Note that the sum from $j = i + 2$ to $j = k^+$ is 0 when $i + 2 > k^+$ in the first line of the above equation. In a similar way, the expected fraction of redundant nodes is given by

$$n_{\rm red} = \sum_{k^-=0}^{\infty}\sum_{k^+=k^-+1}^{\infty} P_{k^-}P_{k^+}\left[\sum_{i=0}^{k^-} C_{k^-}^{i} P_d^i (1-P_d)^{k^--i}\left(\sum_{j=0}^{i} C_{k^+}^{j} P_b^j (1-P_b)^{k^+-j}\right)\right]$$
$$+ \sum_{k^-=k^+=1}^{\infty} P_{k^-}P_{k^+}\left[\sum_{i=0}^{k^+-1} C_{k^+}^{i} P_b^i (1-P_b)^{k^+-i}\left(\sum_{j=i+1}^{k^-} C_{k^-}^{j} P_d^j (1-P_d)^{k^--j}\right)\right]$$
$$+ \sum_{k^+=0}^{\infty}\sum_{k^-=k^++1}^{\infty} P_{k^-}P_{k^+}\left[\sum_{i=0}^{k^+} C_{k^+}^{i} P_b^i (1-P_b)^{k^+-i}\left(\sum_{j=i+1}^{k^-} C_{k^-}^{j} P_d^j (1-P_d)^{k^--j}\right)\right]. \qquad (2)$$

In all other cases, the node is ordinary. So the expected fraction of ordinary nodes is described by

$$n_{\rm ord} = 1 - n_{\rm crit} - n_{\rm red}. \qquad (3)$$

For model networks, the expected fraction of critical or redundant nodes with different disturbance strengths can be described by analytical formulas. To be specific, the disturbance strength of a node is determined by the in and out degrees of itself, the number of weakly divergent nodes in its upstream neighbors, and the number of balanced nodes in its downstream neighbors. By enumerating all possible combinations, we can offer analytical formulas for the expected fraction of the critical nodes with disturbance strength δ, that is

$$n_{\rm crit}^{\delta} = \sum_{k^-=0}^{\infty}\sum_{k^+=k^-+1}^{\infty} P_{k^-}P_{k^+}\left[\sum_{i=0}^{k^-} C_{k^-}^{i} P_d^i (1-P_d)^{k^--i} C_{k^+}^{i+\delta+1} P_b^{i+\delta+1}(1-P_b)^{k^+-i-\delta-1}\right]$$
$$+ \sum_{k^-=k^+=\delta}^{\infty} P_{k^-}P_{k^+}\left[\sum_{i=0}^{k^--\delta} C_{k^-}^{i} P_d^i (1-P_d)^{k^--i} C_{k^+}^{i+\delta} P_b^{i+\delta}(1-P_b)^{k^+-i-\delta}\right]$$
$$+ \sum_{k^+=\delta}^{\infty}\sum_{k^-=k^++1}^{\infty} P_{k^-}P_{k^+}\left[\sum_{i=\delta}^{k^+} C_{k^+}^{i} P_b^i (1-P_b)^{k^+-i} C_{k^-}^{i-\delta} P_d^{i-\delta}(1-P_d)^{k^--i+\delta}\right]. \qquad (4)$$

Note that $C_{k^+}^{i+\delta+1} = 0$ when $i + \delta + 1 > k^+$ in the first line of the above equation. In a similar way, the expected fraction of the redundant nodes with disturbance strength δ is given by

$$n_{\rm red}^{\delta} = \sum_{k^-=\delta-1}^{\infty}\sum_{k^+=k^-+1}^{\infty} P_{k^-}P_{k^+}\left[\sum_{i=\delta-1}^{k^-} C_{k^-}^{i} P_d^i (1-P_d)^{k^--i} C_{k^+}^{i-\delta+1} P_b^{i-\delta+1}(1-P_b)^{k^+-i+\delta-1}\right]$$
$$+ \sum_{k=\delta}^{\infty} P_{k^-}P_{k^+}\left[\sum_{i=0}^{k^+-\delta} C_{k^+}^{i} P_b^i (1-P_b)^{k^+-i} C_{k^-}^{i+\delta} P_d^{i+\delta}(1-P_d)^{k^--i-\delta}\right]$$
$$+ \sum_{k^+=0}^{\infty}\sum_{k^-=k^++\delta}^{\infty} P_{k^-}P_{k^+}\left[\sum_{i=0}^{k^+} C_{k^+}^{i} P_b^i (1-P_b)^{k^+-i} C_{k^-}^{i+\delta} P_d^{i+\delta}(1-P_d)^{k^--i-\delta}\right]. \qquad (5)$$

By inserting the degree distribution of different model networks into the general formulas, analytical results of the nodes with different categories and disturbance strength can be derived. Detailed analytical results are presented below.

For an ER network, obviously, the in and out degrees follow a Poisson distribution, that is

$$P(k_v^+ = k) = P(k_v^- = k) = \frac{\langle k\rangle^k e^{-\langle k\rangle}}{k!}. \qquad (6)$$

By substituting the Poisson distribution into the probability formula of the weakly divergent node, we obtain

$$P_{\rm d} = \sum_{k=0}^{\infty}\frac{\langle k\rangle^k e^{-\langle k\rangle}}{k!}\frac{\langle k\rangle^{k+1} e^{-\langle k\rangle}}{(k+1)!} = I_1(2\langle k\rangle)e^{-2\langle k\rangle}, \qquad (7)$$

where $I_a(x)$ is the modified Bessel function. Similarly, the probability of the balanced node is given by

$$P_{\rm b} = \sum_{k=1}^{\infty} \frac{\langle k\rangle^k e^{-\langle k\rangle}}{k!}\frac{\langle k\rangle^k e^{-\langle k\rangle}}{k!} = I_0(2\langle k\rangle)e^{-2\langle k\rangle} - e^{-2\langle k\rangle}. \quad (8)$$

For EX network, obviously, the in and out degrees follow an exponential distribution, that is

$$P(k_v^+ = k) = P(k_v^- = k) = Ce^{-k/\kappa}, \quad (9)$$

where $C = 1 - e^{-1/\kappa}$ and $\kappa = 1/\log\frac{1+\langle k\rangle}{\langle k\rangle}$. By substituting the exponential distribution into the probability formula of the weakly divergent node, we obtain

$$P_{\rm d} = \sum_{k=0}^{\infty} Ce^{-k/\kappa}Ce^{-(k+1)/\kappa} = \frac{\langle k\rangle}{(\langle k\rangle+1)(2\langle k\rangle+1)}. \quad (10)$$

Similarly, the probability of the balanced node is given by

$$P_{\rm b} = \sum_{k=1}^{\infty} Ce^{-k/\kappa}Ce^{-k/\kappa} = \frac{\langle k\rangle^2}{(\langle k\rangle+1)^3 + \langle k\rangle(\langle k\rangle+1)^2}. \quad (11)$$

For the SF network, obviously, the in and out degrees follow a power-law distribution with scaling exponent γ and exponential cutoff parameter κ, that is,

$$P(k_v^+ = k) = P(k_v^- = k) = Ck^{-\gamma}e^{-k/\kappa}, \quad (12)$$

where $C = 1/{\rm Li}_\gamma(e^{-1/\kappa})$, $\langle k\rangle = C{\rm Li}_{\gamma-1}(e^{-1/\kappa})$, $P_0 = 0$, and ${\rm Li}_s(z)$ is the polylogarithm function. By substituting the power-law distribution into the probability formula of the weakly divergent node, we obtain

$$P_{\rm d} = \sum_{k=1}^{\infty} Ck^{-\gamma}e^{-k/\kappa}C(k+1)^{-\gamma}e^{-(k+1)/\kappa} = \frac{e^{-1/\kappa}}{{\rm Li}_\gamma(e^{-1/\kappa})^2}\sum_{k=1}^{\infty}\frac{e^{-2k/\kappa}}{(k^2+k)^\gamma}. \quad (13)$$

When $\kappa \to \infty$, the exponential cutoff vanishes and the polylogarithm function reduces to the Riemann ζ function $\zeta(s)$. Thus, $P_{\rm d}$ becomes

$$P_{\rm d} = \frac{1}{\zeta(\gamma)^2}\sum_{k=1}^{\infty}\frac{1}{(k^2+k)^\gamma}. \quad (14)$$

Similarly, the probability of the balanced node is given by

$$P_{\rm b} = \sum_{k=1}^{\infty} Ck^{-\gamma}e^{-k/\kappa}Ck^{-\gamma}e^{-k/\kappa} = \frac{{\rm Li}_{2\gamma}(e^{-2/\kappa})}{{\rm Li}_\gamma(e^{-1/\kappa})^2}. \quad (15)$$

When $\kappa \to \infty$, $P_{\rm b}$ becomes

$$P_{\rm b} = \frac{\zeta(2\gamma)}{\zeta(\gamma)^2}. \quad (16)$$

IV. DEGREE CORRELATIONS

We assume that the in and out degrees of nodes have no correlation in the analysis, which is true for all of the model networks studied above. However, the in- and out-degree correlation exists in real-world networks [25,26]. It is unreasonable to assume that such correlation has no influence on the robustness of control. Thus, we use the Pearson correlation coefficient [27] to quantify the in- and out-degree correlation of a network and to analyze the effect of the correlation on the robustness of control.

A. In- and out-degree correlation

The Pearson correlation coefficient, which is used to quantify the in- and out-degree correlation of a network, is described by

$$R = \frac{\sum_{i=1}^N (k_i^- - \overline{k^-})(k_i^+ - \overline{k^+})}{\sqrt{\sum_{i=1}^N (k_i^- - \overline{k^-})^2}\sqrt{\sum_{i=1}^N (k_i^+ - \overline{k^+})^2}}, \quad (17)$$

where k_i^- and k_i^+ are the in degree and out degree of node i, respectively. $\overline{k^-} = (1/N)\sum_{i=1}^N k_i^-$ and $\overline{k^+} = (1/N)\sum_{i=1}^N k_i^+$ are the average in degree and average out degree, and $R \in [-1,1]$.

For a model network generated by the same in-degree sequence $K_{\rm in} = \{k_1^-, k_2^-, \ldots, k_N^-\}$ and out-degree sequence $K_{\rm out} = \{k_1^+, k_2^+, \ldots, k_N^+\}$ in ascending order, the adjusting strategy of its in- and out-degree correlation consists three steps. First, we assign each isolated node with in degree k_i^- and out degree k_{N-i+1}^+. In this case, the generated network has the strongest negative in- and out-degree correlation; i.e., a node with smaller in degree (out degree) has bigger out degree (in degree). Note that the correlation coefficient R may not reach -1 for some in- and out-degree sequences. Second, we increase the value of R by randomly exchanging the out degrees of nodes. This process will be continued until $R \approx 0$; i.e., the generated network has no in- and out-degree correlation. Finally, we increase the value of R by assigning more nodes with the same in and out degrees. This process will be continued until $R = 1$, i.e., the generated network, in which all nodes are balanced, has the strongest positive in- and out-degree correlation.

B. The effect of in- and out-degree correlation

We perform simulations on ER, EX, and SF networks with varied correlation coefficient R. The simulation results in Fig. 5 clearly show a general trend: The excessive increase of the positive correlation weakens the robustness of control. An extreme case is that a network has the strongest positive correlation between in- and out-degrees ($R = 1$), i.e., all nodes of the network are balanced. In this case, each node with $k^+ > 0$ is critical since there are no weakly divergent nodes in its upstream neighbors while all nodes in its downstream neighbors are balanced. Meanwhile, the disturbance strength of a critical node equals its out degree k^+.

One key result from Ref. [11] is that positively correlated in and out degrees enhance the controllability of the edge dynamics by decreasing the fraction of driver nodes. The reason is that more balanced nodes appear in the network with the enhancing of the positive correlation between in and out degrees. A balanced node represents the transport process mapping an input space into a similarly dimensional output space. Its balanced state will be broken by the dimensionality

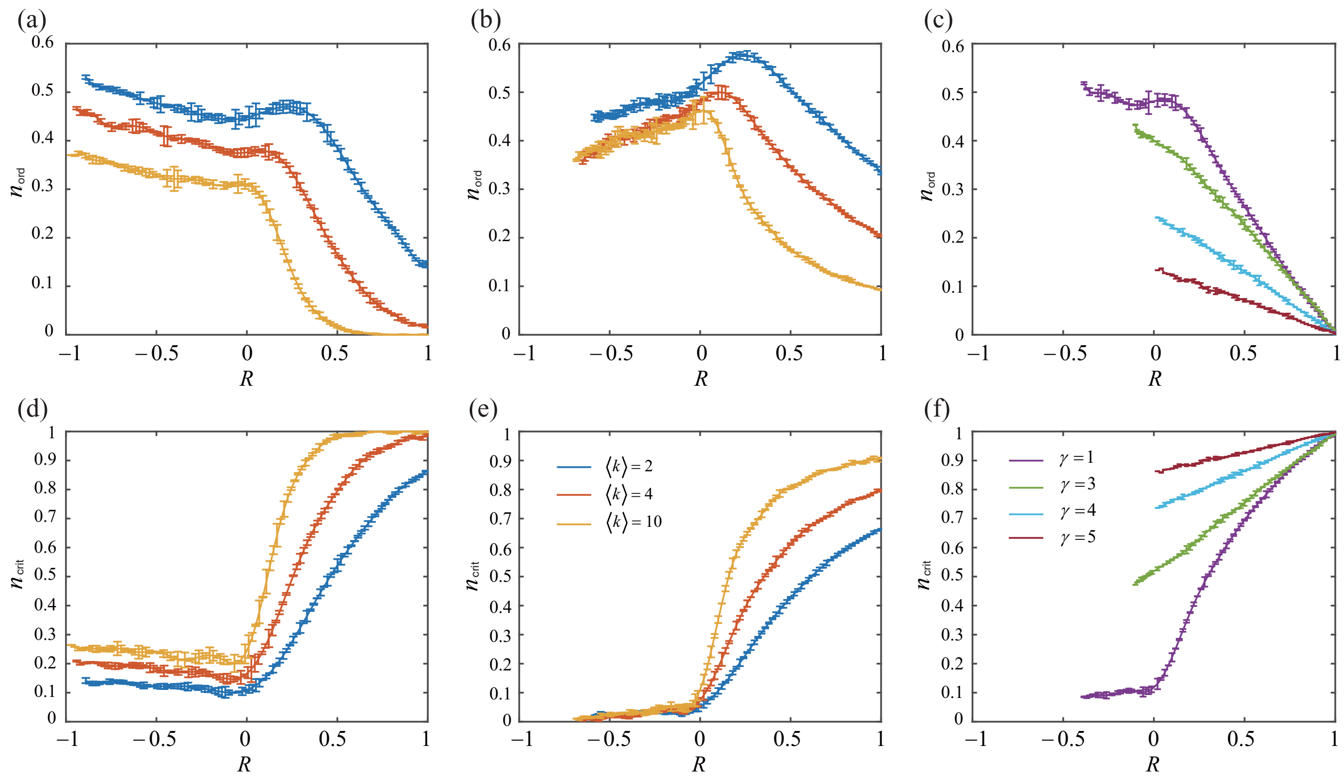

FIG. 5. Effect of degree correlations. Fraction of ordinary and critical nodes in ER network [(a), (d)], EX network [(b), (e)], and SF network [(c), (f)] as the function of the correlation coefficient R. In SF network, the exponential cutoff parameter $\kappa = 10$. The average degree for the lines in panels (a) and (b) are, from top to bottom, $\langle k \rangle = 2,4,10$, and reverse order for panels (d) and (e). The scale-free exponent for the lines in panel (c) are, from top to bottom, $\gamma = 1,3,4,5$, and reverse order for panel (f). All the simulation results are obtained by averaging over 30 independent networks realizations.

reduction of its input space. Thus, a network with more balanced nodes has better controllability properties but has weaker robustness of control against node failure.

V. OPTIMIZING

The optimal robustness defines the optimizing situation, where the removal of any node in a network will not change the number of driver nodes required to maintain full controllability. In other words, all nodes are ordinary in the network. In order to realize the optimal robustness of controlling edge dynamics, adding a circuit-link strategy is proposed in this section. For each edge e_{uv} in a given network, we add its circuit link e_{vu}. In this case, a node u in the upstream or downstream neighbors of a removed node v will lose one incoming edge e_{vu} and one outgoing edge e_{uv} at the same time, and hence its category will not change. Note that, in this case, all nodes are balanced. However, there are two directed edges with opposite directions between any pair of connected nodes, which is fundamentally different from a common network

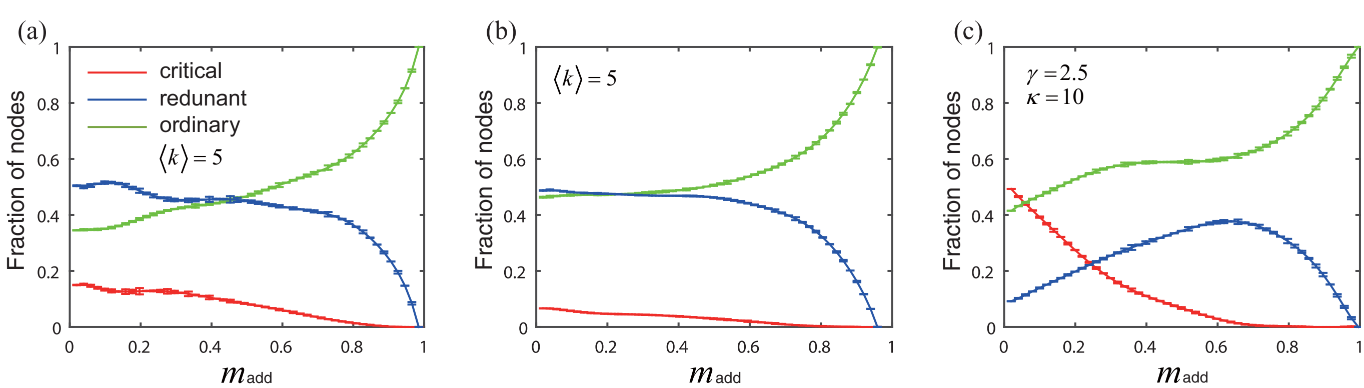

FIG. 6. Optimizing. Fraction of critical, redundant, and ordinary nodes in ER network (a), EX network (b), and SF network (c) as the function of the proportion of added edges m_{add}. The lines are, from top to bottom, the fraction of ordinary, redundant, and critical nodes. All the simulation results are obtained by averaging over 30 independent network realizations.

with the strongest positive correlation between in and out degrees.

The empirical evidence of the circuit-link strategy is provided in Fig. 6. We show the fractions of the three categories of nodes in model networks as the function of the proportion of added edges $m_{\text{add}} = M_{\text{add}}/M$. One can see that the fraction of the ordinary nodes increases monotonously with the increase of m_{add}. The simulation results indicate that the circuit-link strategy can effectively enhance the control robustness. To some extent, the circuit-link strategy could take a guiding role in optimizing the control robustness against node failure.

VI. SUMMARY

By classifying each node according to the change in the number of driver nodes when the node and its links are removed, we quantified the robustness of controlling edge dynamics in networks against node failure. To be specific, a network with more ordinary nodes has higher degree of robustness against changes in network structure due to random node failure. Based on the simulation and analysis results in model and real-world networks, we found that the percentages of the three types of nodes are mainly related to the degree distribution of networks. In other words, the robustness of controlling edge dynamics is, to a great extent, encoded by the degree distribution of networks. The results deepen our understanding of the robustness of controlling edge dynamics in networks.

ACKNOWLEDGMENTS

S.-P.P. and F.H. are supported by NSFC under Grants No. 61573036 and No. 61174057, BNSF under Grant No. 4112034. W.-X.W. was supported by NSFC under Grant No. 11105011, CNNSF under Grant No. 61074116, and the Fundamental Research Funds for the Central Universities.

APPENDIX: REAL-WORLD NETWORKS

The details of the real-world networks we have studied are presented in Table I.

TABLE I. Summary of the real-world networks analyzed in the paper. For each network, we show the number of nodes N, the number of edges M, the fraction of critical nodes n_{crit}, the fraction of ordinary nodes n_{ord}, and the fraction of redundant nodes n_{red}.

Type	No.	Name	N	M	n_{crit}	n_{ord}	n_{red}
Regulatory	1	Ownership-USCorp [28]	8497	6726	0.002	0.790	0.208
	2	TRN-EC-2 [24]	423	578	0.007	0.695	0.298
	3	TRN-Yeast-1 [29]	4684	15 451	0.001	0.944	0.055
	4	TRN-Yeast-2 [24]	688	1079	0.001	0.781	0.218
Trust	5	Prison inmate [30,31]	67	182	0.060	0.567	0.373
	6	WikiVote [32]	7115	1 03 689	0.004	0.709	0.287
Food Web	7	St. Marks [33]	45	224	0	0.289	0.711
	8	Seagrass [34]	49	226	0.041	0.347	0.612
	9	Grassland [35]	88	137	0.045	0.625	0.330
	10	Ythan [35]	135	601	0.022	0.496	0.482
	11	Silwood [36]	154	370	0.006	0.799	0.195
	12	Little Rock [37]	183	2494	0.082	0.317	0.601
Electronic circuits	13	S208a [24]	122	189	0.057	0.197	0.746
	14	s420a [24]	252	399	0.052	0.198	0.750
	15	s838a [24]	512	819	0.049	0.207	0.744
Neuronal	16	*C. elegans* [38]	297	2359	0.027	0.300	0.673
Citation	17	Small World [39]	233	1988	0.017	0.691	0.292
	18	SciMet [39]	2729	10 416	0.034	0.572	0.394
	19	Kohonen [40]	3772	12 731	0.025	0.684	0.291
World Wide Web	20	Political blogs [41]	1224	19 090	0.016	0.327	0.657
Internet	21	p2p-1 [42,43]	10 876	39 994	0.023	0.673	0.304
	22	p2p-2 [42,43]	8846	31 839	0.013	0.657	0.330
	23	p2p-3 [42,43]	8717	31 525	0.013	0.655	0.332
Organizational	24	Freeman-1 [44]	34	695	0.206	0.441	0.353
	25	Consulting [45]	46	879	0.087	0.413	0.500
Language	26	English words [46]	7381	46 281	0.106	0.712	0.182
	27	French words [46]	8325	24 295	0.139	0.704	0.157
Transportation	28	USair97 [47]	332	2126	0.069	0.545	0.386

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